increase and, as a result, the position hit onto the set $\zeta_2 = 0$ $(\zeta_1 = 0)$.

The author has not succeeded in resolving the question of the existence of controls of the first player increasing the lesser maximum. Therefore, the theorem contains reservations. The difficulty described is sufficiently typical. Its existence and ways for overcoming it were noted in [2].

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ASYMPTOTIC SOLUTION OF CERTAIN PROBLEMS OF TIME-OPTIMAL TYPE

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We examine the optimal control problem for a system in which the process termination time is not fixed. The system of equations of motion contains a small parameter and is reduced to the form of systems with rotating phase. We assume that the frequency depends upon "slow time", while the control occurs in the small terms, so that the system is weakly controllable. Using the averaging method we construct a solution of the optimal control problem and we assume that the time interval over which the process evolves is a quantity of the order of $1/\varepsilon$, where ε is the small parameter. This assumption proves to be a natural one if the terminal manifold depends only on the slow variables. Thus, we investigate the cases, of interest in practice, of controlled systems with small but protracted controls. We solve certain concrete problems with the use of the canonic averaging method developed.

1. Statement of the problem. Let the system's motion be described by the equations

$$a^{\cdot} = \varepsilon f(\tau, a, \psi, u, \varepsilon), \quad a(t_0) = a_0 \tag{1.1}$$

$$\psi^{\cdot} = \nu(\tau) + \varepsilon F(\tau, a, \psi, u, \varepsilon), \quad \psi(t_0) = \psi_0$$

Here a is the n-dimensional vector of "slow" variables, belonging to some bounded domain of definition and smoothness of the functions f and F; $\tau = \varepsilon (t - t_0) + \tau_0$ is slow time; ψ is a scalar "fast" variable (the phase) varying on an unbounded interval; ε is a small parameter, $\varepsilon \in [0, \varepsilon_0]$; t_0, τ_0, a_0 and ψ_0 are initial data, t is time, $t \in [t_0, t_1]$, $t_1 \sim 1 / \varepsilon$. The m-dimensional control vector $u(t)$ is assumed to belong to the class of piecewise-continuous functions such that the solution of system (1.1) exists and is unique on the interval being considered. The functions f, ν and F are assumed sufficiently smooth in the slow variables; moreover, $\nu(\tau) \geqslant \nu_0 > 0$. The functions f and E are assumed periodic with period 2π and piecewise continuous with respect to the fast phase ψ. The requirement of smoothness with respect to the slow variables is dictated by the order of the approximations (in powers of ε). By analogy with uncontrollable systems [1 – 3] Eqs. (1.1) can be called a weakly-controllable system with rotating phase. A number of oscillatory systems subject to perturbations and small controls reduce to form (1.1).

We pose the following optimal control problem. Find a piecewise-continuous control $u(t)$ such that at some unfixed instant t_1 the slow phase variables would belong to manifold (1.2) and, here, the minimal of functional (1.3) (g is a scalar function) would be achieved

$$G(\tau, a, \varepsilon)|_{t_1} = 0, \qquad G = (G_1, \ldots, G_l), \qquad 1 \leqslant l \leqslant n. \tag{1.2}$$

$$J = g(\tau_1, a(t_1), \varepsilon) \to \min_{u \in U}, \qquad \tau_1 = \varepsilon(t_1 - t_0) + \tau_0 \tag{1.3}$$

The functions G and g are assumed sufficiently smooth. Note that the functional

$$J = g_1(\tau_1, a(t_1), \varepsilon) + \varepsilon \int_{t_0}^{t_1} g_2(\tau, a, \psi, u, \varepsilon)\, dt \tag{1.4}$$

is reduced to the form (1.3). In fact, let us introduce an additional slow variable a_{n+1} changing in accord with the equation

$$a_{n+1} = \varepsilon g_2(\tau, a, \psi, u, \varepsilon), \qquad a_{n+1}(t_0) = 0$$

then functional (1.4) takes on a particular form of (1.3): $J = g_1(\tau_1, a(t_1), \varepsilon) + a_{n+1}(t_1)$.

If manifold (1.2) has the form $a(t_1) = a_1$, where a_1 are specified quantities, while the functional $y = \varepsilon t_1 \to \min$ with respect to $u \in U$, then we obtain the time-optimality problem in the slow variables. We note that in problems with small controls the value of the fast variable, namely, the phase ψ, is usually not fixed, i.e. its first endpoint is free. The cases when it is necessary to allow for a dependency on the fast variable ψ in (1.2), (1.3), as well as when the initial manifold is

$$L(\tau, a, \psi, \varepsilon)|_{t_0} = 0, \qquad L = (L_1, \ldots, L_r), \quad 1 \leqslant r \leqslant n$$

require a special consideration.

Let us assume that the optimal problem (1.1) – (1.3) has a unique solution for all sufficiently small values of the parameter $\varepsilon > 0$ being considered here. Then the optimal control and trajectory satisfy the maximum principle [4] which can be stated as follows in the case under examination. Let p be an n-dimensional vector of the variables adjoint to vector a, and let q denote the scalar adjoint variable corresponding to ψ. Then, at any instant $t \in [t_0, t_1]$ the equality

$$H^* \equiv \varepsilon\,(pf\,(\tau,\, a,\, \psi,\, u^*,\, \varepsilon)) + [\nu\,(\tau) + \varepsilon F\,(\tau,\, a,\, \psi,\, u^*,\, \varepsilon)]\; q = \max_{u \in U}\, \mathrm{H} \equiv \max_{u \in U}\, \{\varepsilon(pf\,(\tau,\, a,\, \psi,\, u,\, \varepsilon)) + [\nu\,(\tau) + \varepsilon F\,(\tau,\, a,\, \psi,\, u,\, \varepsilon)]\; q\} \tag{1.5}$$

is valid for the optimal control $u^*\,(t)$. Here

$$H = \varepsilon\,(pf) + (\nu + \varepsilon F) q \equiv \nu q + \varepsilon h \tag{1.6}$$

is the Hamiltonian function of the system, (pf) denotes the scalar product of vectors p and f, while p and q satisfy the adjoint equations and the transversality conditions at the right endpoint (1.7)

$$p^{\cdot} = -\,\varepsilon \frac{\partial\,(pf)}{\partial a}\Big|_{u^*} - \varepsilon q \frac{\partial F}{\partial a}\Big|_{u^*}, \quad p\,(t_1) = -\,\frac{\partial g}{\partial a}\Big|_{t_1} - \frac{\partial\,(\alpha G)}{\partial a}\Big|_{t_1}, \qquad \alpha = (\alpha_1, \ldots, \alpha_l)$$

$$q^{\cdot} = -\,\varepsilon \frac{\partial\,(pf)}{\partial \psi}\Big|_{u^*} - \varepsilon q \frac{\partial F}{\partial \psi}\Big|_{u^*}, \quad q\,(t_1) = 0$$

The variables a and ψ satisfy system (1.1) with $u = u^*\,(t)$ and the equality

$$H^*\big|_{t_1} = \varepsilon \frac{\partial g}{\partial \tau}\Big|_{t_1} + \varepsilon\Big(\alpha\, \frac{\partial G}{\partial \tau}\Big)\Big|_{t_1} \tag{1.8}$$

must be fulfilled at the interval's endpoint, while

$$H^*\,(t) = H^*\,(t_1) = 0, \qquad t \in [t_0,\ t_1] \tag{1.9}$$

if there is no dependency on slow time. If, for example, the constraints on the control are removed, i.e. $u \in R_m$, then, as a consequence of assuming the functions f and F to be smooth, the necessary condition for the maximality of H with respect to u in (1.5) with the other arguments fixed becomes

$$\partial H \,/\, \partial u_i = 0, \; i = 1, 2, \ldots, m \tag{1.10}$$

from which the desired optimal control can be determined as a function of τ, a, ψ, p, q and ε

$$u^* = V\,(\tau,\, a,\, \psi,\, p,\, q,\, \varepsilon) \tag{1.11}$$

Here the function V is assumed sufficiently smooth and periodic in ψ with period 2π. We assume that Eq. (1.10) is uniquely solvable relative to the components of vector u and that u^* is really the maximum point. These conditions are fulfilled, obviously, if the function H is strictly upper convex, for example, is a negative definite quadratic form in u. However, if the controls are subject to certain geometric constraints, then the optimal control (1.11) is determined from the general condition (1.5) and its determination is assumed to be single-valued, i.e. singular controls are absent. Thus, suppose that a sufficiently smooth control V from (1.11) has been determined and substituted into Eqs. (1.1), (1.6) whose right-hand sides are assumed to be functions of ψ, smooth relative to τ, a, p, q, ε and piecewise continuous and periodic with period 2π. Then among the solutions of the resulting boundary value problem (1.1), (1.2), (1.7), (1.8), (1.11) we can find one which is optimal in the sense of (1.3), whose substitution into (1.11) results in a solution of the original optimal control problem (1.1)–(1.3). If the boundary value problem's solution is unique, it determines the solution of the optimal control problem in [4]. However, by a series of examples and by general reasoning we can show that the solution of the boundary value problem, or, more precisely, the solution of the transcendental Eq. (1.8) or (1.9) with respect to t_1, is not unique as a rule. Among the solutions mentioned we select an optimal one giving a minimum to functional (1.3). Such a solution exists by virtue of the assumptions on the existence and uniqueness of the solution of the optimal control problem (1.1)–(1.3).

We should note that small parameter methods were applied in [3, 5–9]. Cases of asymptotically large fixed instant of termination of the control process $T \sim 1/\varepsilon$ were investigated in [8, 9] by the averaging method; asymptotic methods were employed in [3].

2. Construction of canonic averaging of the system. Approximate solution of the problem. System (1.1), (1.7), (1.11) is a standard system with rotating phase, to which the averaging method with respect to the fast variable, i.e. phase ψ, is applicable [1–3]. If Hamiltonian (1.5) is a sufficiently smooth function of the slow variables, the averaged system can be constructed with any degree of accuracy with respect to the small parameter ε. Below, on the basis of the canonic averaging method developed in [9], we construct a new (averaged) Hamiltonian not containing the fast variable. Thus, we construct a univalent canonic change of original variables (a, ψ, p, q) to new (averaged) variables $(\xi, \varphi, \eta, \beta)$ such that, firstly, the corresponding system of equations does not contain the fast variable φ in the right-hand side and, secondly, the original and the new variables coincide when $\varepsilon = 0$. Such a change is effected by a generating function S [10]

$$p = \frac{\partial S}{\partial a}, \quad q = \frac{\partial S}{\partial \psi}, \quad \xi = \frac{\partial S}{\partial \eta}, \quad \varphi = \frac{\partial S}{\partial \beta} \quad (S = S(\tau, a, \psi, \eta, \beta, \varepsilon)) \quad (2.1)$$

where S must yield the identity transformation when $\varepsilon = 0$, i.e.

$$S = (a\eta) + \psi\beta + \varepsilon\sigma(\tau, a, \psi, \eta, \beta, \varepsilon) \quad (2.2)$$

Consequently,

$$p = \eta + \varepsilon\frac{\partial \sigma}{\partial a}, \quad q = \beta + \varepsilon\frac{\partial \sigma}{\partial \psi}, \quad \xi = a + \varepsilon\frac{\partial \sigma}{\partial \eta}, \quad \varphi = \psi + \varepsilon\frac{\partial \sigma}{\partial \beta} \quad (2.3)$$

Since the original and the new Hamiltonians also must coincide when $\varepsilon = 0$, we seek the averaged Hamiltonian K in the form

$$K = K(\tau, \xi, \eta, \beta, \varepsilon) = \nu(\tau)\beta + \varepsilon k(\tau, \xi, \eta, \beta, \varepsilon) \quad (2.4)$$

The functions H, S and K are subject to the following differential relation

$$\frac{\partial S}{\partial t} + H^*\left(\tau, a, \psi, \frac{\partial S}{\partial a}, \frac{\partial S}{\partial \psi}, \varepsilon\right) = K(\tau, \xi, \eta, \beta, \varepsilon) \quad (2.5)$$

This equation permits us to determine the unknown functions with the accuracy needed. With due regard to (2.3), by substituting the representations (2.2) and (2.4), we transform the equation obtained

$$\nu\frac{\partial \sigma}{\partial \psi} + h\left(\tau, a, \psi, \eta + \varepsilon\frac{\partial \sigma}{\partial a}, \beta + \varepsilon\frac{\partial \sigma}{\partial \psi}, \varepsilon\right) + \varepsilon\frac{\partial \sigma}{\partial \tau} = k\left(a + \varepsilon\frac{\partial \sigma}{\partial \eta}, \eta, \beta, \varepsilon\right) \quad (2.6)$$

where $h = H^* - q\nu$. We construct the solution of Eq. (2.6) approximately as series expansions in integer powers of the small parameter

$$\sigma = \sigma_0(\tau, a, \psi, \eta, \beta) + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \ldots, \quad (2.7)$$
$$k = k_0(\tau, \xi, \eta, \beta) + \varepsilon k_1 + \varepsilon^2 k_2 + \ldots$$

We substitute series (2.7) into (2.6), expand with respect to the small parameter, and compare the coefficients of like powers of ε. We obtain a sequence of linking equations the solving of which allows us to find the unknown functions σ_i, k_i. The representations

$$k_i(\tau, a, \eta, \beta) = \frac{1}{2\pi}\int_0^{2\pi} h_i(\tau, a, \psi, \eta, \beta)\, d\psi \equiv \langle h_i \rangle \tag{2.8}$$

$$\sigma_i(\tau, a, \psi, \eta, \beta) = -\frac{1}{\nu(\tau)}\int (h_i - \langle h_i \rangle)\, d\psi, \quad i = 0, 1, 2, \ldots$$

are valid for the coefficients of expansions (2.7). The coefficients h_i are determined sequentially in terms of known quantities. For example, the first three have the form

$$h_0(\tau, a, \psi, \eta, \beta) = h(\tau, a, \psi, \eta, \beta, 0) \tag{2.9}$$

$$h_1(\tau, a, \psi, \eta, \beta) = \frac{\partial\sigma_0}{\partial\tau} + \frac{\partial h_0}{\partial\eta}\frac{\partial\sigma_0}{\partial a} + \frac{\partial h_0}{\partial\beta}\frac{\partial\sigma_0}{\partial\psi} + \left(\frac{\partial h}{\partial\varepsilon}\right)_0 - \frac{\partial k_0}{\partial a}\frac{\partial\sigma_0}{\partial\eta}$$

$$h_2(\tau, a, \psi, \eta, \beta) = \frac{\partial\sigma_1}{\partial\tau} + \frac{\partial h_0}{\partial\eta}\frac{\partial\sigma_1}{\partial a} + \frac{1}{2}\frac{\partial^2 h_0}{\partial\eta^2}\left(\frac{\partial\sigma_0}{\partial a}\right)^2 + \frac{\partial h_0}{\partial\beta}\frac{\partial\sigma_1}{\partial\psi} +$$
$$\frac{1}{2}\frac{\partial^2 h_0}{\partial\beta^2}\left(\frac{\partial\sigma_0}{\partial\psi}\right)^2 + \frac{\partial^2 h_0}{\partial\eta\,\partial\beta}\frac{\partial\sigma_0}{\partial a}\frac{\partial\sigma_0}{\partial\psi} + \frac{\partial^2 h_0}{\partial\eta\,\partial\varepsilon}\frac{\partial\sigma_0}{\partial a} + \frac{\partial^2 h_0}{\partial\beta\,\partial\varepsilon}\frac{\partial\sigma_0}{\partial\psi} +$$
$$\frac{1}{2}\left(\frac{\partial^2 h}{\partial\varepsilon^2}\right)_0 - \frac{\partial k_0}{\partial a}\frac{\partial\sigma_1}{\partial\eta} - \frac{1}{2}\frac{\partial^2 k_0}{\partial a^2}\left(\frac{\partial\sigma_0}{\partial\eta}\right)^2 - \frac{\partial k_1}{\partial a}\frac{\partial\sigma_0}{\partial\eta}$$

and so on. From (2.9), (2.8) it follows that the number of coefficients is delimited by the degree of smoothness of the original Hamiltonian.

If the right-hand sides of system (1.1), (1.7) with $u = V$ are only piecewise-continuous in ψ, then we can write down the so-called first-approximation system and the corresponding initial and boundary conditions [11]

$$\xi^{\cdot} = \frac{\partial K}{\partial\eta} = \varepsilon\langle f_0^*(\tau, \xi, \eta, \beta)\rangle, \quad \xi(t_0) = a_0, G_0(\tau_1, \xi(t_1)) = 0 \tag{2.10}$$
$$\eta^{\cdot} = -\frac{\partial K}{\partial\xi} = -\varepsilon\frac{\partial}{\partial\xi}(\langle f_0^*\rangle\,\eta), \quad \eta(t_1) = -g'_{0\xi}|_{t_1} - (\alpha G'_{0\xi})_{t_1}$$
$$\varphi^{\cdot} = \frac{\partial K}{\partial\beta} = \nu(\tau) + \varepsilon\langle F_0^*(\tau, \xi, \eta, \beta)\rangle, \quad \varphi(t_0) = \psi_0$$
$$\beta = \text{const}, \ \beta(t_1) = \beta = 0$$

Here and in (2.8) the angle brackets denote averaging with respect to ψ ; for example,

$$\langle f_0^*(\tau, \xi, \eta, \beta)\rangle = \frac{1}{2\pi}\int_0^{2\pi} f^*(\tau, \xi, \psi, \eta, \beta, 0)\, d\psi$$

while an expression of the type (XY) denotes, as before, the scalar product of vectors X and Y.

In the boundary-value problem of first approximation the first two vector equations are integrable independently of φ, which permits us to introduce the slow time $s = \varepsilon(t - t_0)$

$$\frac{d\xi}{ds} = f_0(\tau, \xi, \eta), \quad \xi(0) = a_0, \quad G_0(\tau_1, \xi(s_1)) = 0 \tag{2.11}$$
$$\frac{d\eta}{ds} = -\frac{\partial}{\partial\xi}(f_0(\tau, \xi, \eta)\,\eta), \quad \eta(s_1) = -g'_{0\xi}(\tau_1, \xi(s_1)) - (\alpha G'_{0\xi}(\tau_1, \xi(s_1)))$$

Here $s_1 = \varepsilon(t_1 - t_0)$, while, as a consequence of the boundary conditions, the constant β is assumed equal to zero with an error of the order of ε. In fact, from (2.3) and (1.7) it follows that

$$\beta = -\varepsilon \frac{\partial \sigma}{\partial \psi}\Big|_{t_1} \sim O(\varepsilon)$$

The initial and boundary conditions for the first-approximation problem can be written out, with an error of the order of ε, as equations in the slow time s. We note that in the problem under analysis the averaged phase — the fast variable φ — determines ψ also with an error of the order of ε after an integration of the system of slow variables

$$\psi = \psi_0 + \int_{t_0}^{t} [\nu(\tau') + \varepsilon F_0(\tau', \xi(s'), \eta(s'))]\, dt' + O(\varepsilon) \qquad (2.12)$$

In slow time s we obtain for φ

$$\varphi(s, \varepsilon) = \psi_0 + \frac{1}{\varepsilon}\int_0^s [\nu(\tau') + \varepsilon F_0(\tau', \xi(s'), \eta(s')]\, ds' \qquad (2.13)$$

With regard to the boundary-value problem (2.11) we assume that it admits of a unique solution for any given $s_1 \sim 1$.

Let us now treat condition (1.8) as an equation for determining the control process termination instant. Into this equation we substitute the approximate expressions found

$$a = \xi(s,s_1) + O(\varepsilon), \quad p = \eta(s, \quad s_1) + O(\varepsilon), \quad \beta = O(\varepsilon)$$
$$\psi = \varphi(s,s_1, \varepsilon) + O(\varepsilon)$$

We obtain an approximate relation for determining s_1

$$h(s_1) = (\eta(s_1, s_1) f^*(\tau_1, \xi(s_1, s_1), \varphi(s_1, s_1, \varepsilon), \eta(s_1, s_1), 0, 0) + \qquad (2.14)$$
$$O(\varepsilon) = \partial g_0 / \partial \tau_1 + (\alpha \partial G_0 / \partial \tau_1)$$

The form of the left-hand side establishes that the transcendental equation obtained has, in general, many roots the number of which tends to infinity as $[1 / \varepsilon]$ for $\varepsilon \to 0$ because a rapidly oscillating function of s_1 with a frequency of the order of $1 / \varepsilon$ and an amplitude of the order of unity occurs on the left. The set $\{s_1^*\}$ of roots of this equation forms a discrete interval of length of the order of unity, while the distance between adjacent roots is of the order of ε. The values of the roots from this set are determined with an obligatory error of the order of ε^2. Obviously, the desired optimal solution is obtained by minimizing the approximate value of the functional over the set $\{s_1^*\}$ of admissible roots. Without lessening the accuracy with respect to the slow variables and to the functional we can determine the magnitude of s_1 also with an error of the order of ε. Then the admissible root set $\{s_1^*\}$ is continuous, while a root of the equation occurs at the ε-neighborhood of any value in the sense indicated above.

Let us find the minimal value of the approximate functional and the corresponding value of s_1^*. For this purpose we make use of the expression for the averaged Hamiltonian (2.4), containing the unknown parameter β. We write out this value with an error of the order of ε^2 by using the second formula in (2.3), taken for $s = s_1$, and the corresponding expression for σ_0 (see (2.8) and (2.9)). We obtain

$$\beta = -\varepsilon \frac{\partial \sigma_0}{\partial \psi}\Big|_{s_1} + O(\varepsilon^2) \qquad (2.15)$$

if the value of s_1 is known with error $O(\varepsilon^2)$ as well. We write out the expression for the averaged Hamiltonian

$$K(s_1) = \varepsilon(\eta\langle f_0^*\rangle)_{s_1} + \nu(\tau_1)\beta - \varepsilon\left(\frac{\partial g_0}{\partial\tau_1} + \left(\alpha\frac{\partial G_0}{\partial\tau_1}\right)\right) + O(\varepsilon^2) = 0 \quad (2.16)$$

Substituting here the expression (2.15) for β we obtain the same equation (2.14) for the approximate determination of the unknown parameter s_1 of the problem. Later on, however, we shall determine the magnitude of s_1 with error $O(\varepsilon)$. Setting $\varkappa = \beta / \varepsilon$ and discarding terms $O(\varepsilon)$, from (2.16) we obtain an equation for s_1

$$(\eta\langle f_0^*\rangle)_{s_1} - \frac{\partial g_0}{\partial\tau_1} - \left(\alpha\frac{\partial G_0}{\partial\tau_1}\right) + \nu(\tau_1)\varkappa = 0 \quad (2.17)$$

This equation allows us to determine with error $O(\varepsilon)$ the process termination instant as a function of parameter $\varkappa$ from some continuous interval containing, by virtue of (2.15), the point zero. As established above, the quantity $\varkappa$ should be chosen such that the magnitude of functional (1.3), computed approximately

$$J_0(\varkappa) = g_0(\tau_1(\varkappa),\ \xi(s_1(\varkappa),\ s_1(\varkappa))) \quad (2.18)$$

would reach a minimum with respect to $\varkappa$ (or to $s_1 = s_1(\varkappa)$). The necessary minimum condition is

$$J_0'(\varkappa) = \left(\frac{\partial g_0}{\partial\tau_1} + \frac{\partial g_0}{\partial\xi}\frac{d\xi}{ds_1}\right)\frac{ds_1}{d\varkappa} = 0 \quad (\tau_1 = s_1 + \tau_0) \quad (2.19)$$

Let us assume, further, that using (2.17) we have effected a one-to-one correspondence between s_1 and $\varkappa$ in the domain being examined, i.e.

$$\frac{ds_1}{d\varkappa} = 1 \Big/ \frac{d\varkappa}{ds_1} = \nu^2 \Big/ \left[\nu'\left((\eta f_0) - \frac{\partial g_0}{\partial\tau_1} - \left(\alpha\frac{\partial G_0}{\partial\tau_1}\right)\right) - \nu\frac{d}{ds_1}\left((\eta f_0) - \frac{\partial g_0}{\partial\tau_1} - \left(\alpha\frac{\partial G_0}{\partial\tau_1}\right)\right)\right] \neq 0 \quad (2.20)$$

Then the necessary minimum condition (2.19) reduces to

$$\frac{dg_0}{ds_1} = \frac{\partial g_0}{\partial\tau_1} + \frac{\partial g_0}{\partial\xi}\frac{d\xi}{ds_1} = 0 \quad (2.21)$$

Since according to (2.11)

$$\left.\frac{\partial g_0}{\partial\xi}\right|_{s_1} = -[\eta + (\alpha G_{0\xi}')]_{s_1}$$

the expression

$$\frac{dg_0}{ds_1} = \frac{\partial g_0}{\partial\tau_1} - (\eta f_0)_{s_1} - (\alpha G_{0\xi}')_{s_1}\frac{d\xi}{ds_1}, \qquad \xi = \xi(s_1, s_1)$$

is valid for the unknown derivative. We now substitute the quantity $(\eta f_0)_{s_1}$ in accordance with (2.17) and obtain

$$\frac{\partial g_0}{\partial s_1} = \frac{\partial g_0}{\partial\tau_1} - (\alpha G_{0\xi}')_{s_1}\frac{d\xi}{ds_1} + \left[\nu\varkappa - \frac{\partial g_0}{\partial\tau_1} - \left(\alpha\frac{\partial G_0}{\partial\tau_1}\right)\right]_{s_1} = \left(\alpha\frac{d}{ds_1}G_0(\tau_1, \xi(s_1, s_1))\right) + \nu(\tau_1)\varkappa, \quad \alpha = \alpha(s_1)$$

Since $G_0(\tau_1,\ \xi(s_1,\ s_1)) \equiv 0$ relative to s_1, $dG_0 / ds_1 = 0$. As a result

$$dg_0 / ds_1 = \nu(\tau_1)\varkappa \quad (2.22)$$

whence it follows that

$$J_0'(\varkappa)=\frac{d}{d\varkappa}g_0\big|_{s_1(\varkappa)}=\frac{ds_1}{d\varkappa}\nu(\tau_1)\varkappa,\quad \tau_1=\tau_1(\varkappa) \tag{2.23}$$

Here the functions $s_1(\varkappa)$ and $\tau_1(\varkappa)$ are assumed to have been computed in accord with (2.17). On the basis of the assumptions $\nu\geqslant\nu_0>0$ and (2.20), from (2.23) it follows that the point $\varkappa=0$ is suspected to be at an extremum. If the inequality

$$ds_1\,/\,d\varkappa\,|_{\varkappa=0}>0 \tag{2.24}$$

holds, the value $\varkappa=0$ is a point of local minimum. The condition for the global minimum of point $\varkappa=0$ on the interval being considered of admissible values of $\varkappa$ is

$$\int_0^{\varkappa}\nu(\tau_1(\varkappa'))\frac{ds_1(\varkappa')}{d\varkappa'}\varkappa'd\varkappa'\geqslant 0 \tag{2.25}$$

The inequality obtained can be transformed with the aid of relation (2.17) written as follows :

$$-\chi(s_1(\varkappa))+\nu(\tau_1(\varkappa))\varkappa=0 \tag{2.26}$$

$$\chi(s_1)=\frac{\partial g_0}{\partial\tau_1}+\left(\alpha\frac{\partial G_0}{\partial\tau_1}\right)-(\eta/_0)_{s_1},\quad \tau_1=s_1+\tau_0,\quad \alpha=\alpha(s_1) \tag{2.27}$$

From (2.26) we find $\varkappa=\chi\,/\,\nu$, which permits us to write inequality (2.25) as

$$\int_{s_1(0)}^{s_1}\chi(s_1')\,ds_1'\geqslant 0 \tag{2.28}$$

Here the problem's parameter s_1 belongs to some neighborhood being examined of the point $s_1(0)$, i.e. of the locally optimal value of the process termination instant, corresponding to $\varkappa=0$.

Thus, if inequality (2.28) has been established, then this completes the procedure of constructing the optimal first-approximation solution. The solution algorithm for the optimal control problem (1.1)–(1.3) reduces to solving boundary-value problem (2.11) relative to the $2n$ variables ξ and η on a bounded interval ($s_1\sim 1$) and the optimal control process termination instant is given by relation (2.17) in which $\varkappa=0$. The minimal value of functional (1.3) equals $J_0(0)$ with error $O(\varepsilon)$; the approximate optimal control is obtained from (1.11) by substituting into it the expressions found

$$u_0^*=V(\tau,a,\psi,\eta,0,0)=V(\tau,\xi,\varphi,\eta,0,0)+O(\varepsilon) \tag{2.29}$$

If in the first approximation there is no dependency on τ, the solution algorithm simplifies and reduces to constructing a solution of the boundary-value problem

$$\frac{d\xi}{ds}=f_0(\xi,\eta),\quad \xi(0)=a_0,\ G_0(\xi)\big|_{s_1}=0 \tag{2.30}$$

$$\frac{d\eta}{ds}=-\frac{\partial}{\partial\xi}(f_0(\xi,\eta)\eta),\quad \eta(s_1)=-g_{0\xi}'(\xi)\big|_{s_1}-(\alpha G_{0\xi}')_{s_1}$$

where s_1 is a root of the equation

$$k_0(s_1)=(\eta(s_1,s_1)),\ f_0(\xi(s_1,s_1),\ \eta(s_1,s_1))=0$$

We note that the order of the differential equation system can be lowered, since $k_0(s)=0,\ s\in[0,s_1]$. The solution is found in quadratures for a system with one degree of freedom. Certain examples of the optimal control of quasilinear oscillatory systems with one degree of freedom were solved in the first approximation in Sect. 3.

Let us consider briefly the construction procedure for solutions of higher approximations in ε. For this we must write out the averaged boundary-value problem of zeroth approximation with the aid of formulas (2.7)–(2.9), which also permits us to introduce the slow time $s = \varepsilon (t - t_0)$

$$\frac{d\xi}{ds} = \frac{\partial}{\partial \eta} k(\tau, \xi, \eta, \beta, \varepsilon), \quad \xi(0) = \xi_0 \tag{2.31}$$

$$\frac{d\eta}{ds} = -\frac{\partial}{\partial \xi} k(\tau, \xi, \eta, \beta, \varepsilon), \quad \eta(s_1) = \eta_1$$

$$\frac{d\varphi}{ds} = \frac{\nu(\tau)}{\varepsilon} + \frac{\partial}{\partial \beta} k(\tau, \xi, \eta, \beta, \varepsilon), \quad \varphi(0) = \varphi_0, \quad \beta = \mathrm{const}$$

Here the unknown constants ξ_0, η_1, s_1, φ_0 and β are determined from the initial conditions (1.1) for a and ψ, from the process termination conditions (1.2), from the transversality conditions (1.7) at the right endpoint for the adjoint variables and from the condition that the Hamiltonian equals zero at instant s_1. Since the quantity s_1 is found, generally speaking, ambiguously, to determine it we can use the minimum condition for functional (1.3) written out with the appropriate accuracy.

Let us present the solution algorithm. Suppose that the general solution of system (2.31) has been constructed, depending on s and on ξ_0, η_1, s_1, φ_0, β, ε as if on parameters

$$\xi = \xi(s, \xi_0, \eta_1, s_1, \beta, \varepsilon), \quad \eta = \eta(s, \xi_0, \eta_1, s_1, \beta, \varepsilon) \tag{2.32}$$

$$\varphi = \varphi_0 + \int_0^s \left[\frac{\nu(\tau')}{\varepsilon} + \frac{\partial}{\partial \beta} k(\tau', \xi, \eta, \beta, \varepsilon)\right] ds'$$

We solve the last two of Eqs. (2.3) relative to a and ψ to the needed accuracy in ε

$$a = \xi + \varepsilon A(\tau, \xi, \varphi, \eta, \beta, \varepsilon), \quad \psi = \varphi + \varepsilon \Psi(\tau, \xi, \varphi, \eta, \beta, \varepsilon) \tag{2.33}$$

and we substitute into the first two relations; we obtain

$$p = \eta + \varepsilon P(\tau, \xi, \varphi, \eta, \beta, \varepsilon), \quad q = \beta + \varepsilon Q(\tau, \xi, \varphi, \eta, \beta, \varepsilon) \tag{2.34}$$

$$P = \frac{\partial}{\partial a} \sigma(\tau, \xi + \varepsilon A, \varphi + \varepsilon \Psi, \eta, \beta, \varepsilon),$$

$$Q = \frac{\partial}{\partial \psi} \sigma(\tau, \xi + \varepsilon A, \varphi + \varepsilon \Psi, \eta, \beta, \varepsilon)$$

Here A, Ψ, P, Q are known functions. To construct the first two of them we can use the series expansion method or the successive approximations scheme in powers of ε. As a result, for the determination of the unknown parameters ξ_0, φ_0, η_1, s_1 and β we obtain, with the accuracy needed, the system

$$\xi_0 + \varepsilon A(\tau_0, \xi_0, \varphi_0, \eta(0), \beta, \varepsilon) = a_0 \tag{2.35}$$

$$\varphi_0 + \varepsilon \Psi(\tau_0, \xi_0, \varphi_0, \eta(0), \beta, \varepsilon) = \psi_0$$

$$G(\tau_1, \xi(s_1), \varphi(s_1), \eta_1, \beta, \varepsilon) = 0$$

$$\eta_1 + \varepsilon P(\tau_1, \xi(s_1), \varphi(s_1), \eta_1, \beta, \varepsilon) = -g_a'|_{s_1} - (\alpha G_a')_{s_1}$$
$$\beta = -\varepsilon Q(\tau_1, \xi(s_1), \varphi(s_1), \eta_1, \beta, \varepsilon)$$
$$H^*|_{s_1} = \frac{\partial g}{\partial \tau}\bigg|_{s_1} + \left(\alpha \frac{\partial G}{\partial \tau}\right)_{s_1}$$

As was shown for the first-approximation case, the solution of the last transcendental equation in system (2.35) relative to s_1 is found, in general, nonuniquely with the appropriate accuracy. For the parameter s_1 we obtain a discrete set of values (of the order of $[1 / \varepsilon]$) on an interval of finite length and the distance between successive roots is of the order of ε. For choosing the optimal value we must use the control performance index (1.3) and the quantity s_1 can be chosen from some continuous interval. Obviously, in the case of the higher approximations the admissible values of s_1 form, in general, a discrete set. The optimal value of s_1 is found from the minimum condition of functional (1.3), written out with the appropriate accuracy, by minimization over the discrete set $\{s_1^*\}$

$$J^* = \min_{\{s_1^*\}} g(\tau_1, \xi(s_1, s_1) + \varepsilon A|_{s_1}, \varepsilon) \tag{2.36}$$

where $\{s_1^*\}$ is the set of admissible roots of system (2.35). It is obvious that the optimal value s_1^* lies in an ε-neighborhood of the quantity $s_1(0)$ found from (2.17) with $\varkappa = 0$.

The solution of the optimal control problem (1.1)–(1.3) is constructed analogously to the first-approximation case considered above; the corresponding approximate optimal control is constructed by using expression (1.11) into which we have substituted the solutions (2.34) (and (2.33)). We note that if the vector-valued control functions are subject to certain geometric constraints, then we can work out an analogous procedure for constructing the approximate solution. Since as a rule the right-hand sides are only piecewise-continuous, we can successfully construct only the first approximation. Certain concrete problems are solved in Sect.3 .

3. Examples. Let us investigate in the first approximation certain weakly-controllable oscillatory systems. Suppose that we have a quasilinear controllable system with one degree of freedom

$$x^{\cdot\cdot} + \nu^2(\tau)x = \varepsilon f(\tau, x, x^\cdot, u), \quad x(t_0) = x_0, \quad x^\cdot(t_0) = x_0^\cdot \tag{3.1}$$

Here τ is slow time, x is a coordinate, $x^\cdot$ is the velocity, u is a scalar control, f is some sufficiently smooth function. When $\varepsilon = 0$ the frequency $\nu = \text{const}$, while x and $x^\cdot$ are periodic functions of time

$$x = a\sin\psi, \quad x^\cdot = a\nu\cos\psi, \quad \psi = \nu t + \psi_0 \tag{3.2}$$
$$a = (x_0^2 + x_0^{\cdot 2}/\nu^2)^{1/2} > 0, \quad \psi_0 = \operatorname{arctg} x_0\nu / x_0^\cdot$$

where a is the amplitude of the oscillations, ψ is the phase, ψ_0 is a constant. When $\varepsilon \neq 0$ the system in the new variables a and ψ, connected with the original relations (3.2), is described by equations of type (1.1)

$$a^\cdot = \frac{\varepsilon}{\nu}[f(\tau, a\sin\psi, a\nu\cos\psi, u) - a\nu'\cos\psi]\cos\psi, \qquad a(t_0) = a_0 \tag{3.3}$$
$$\psi^\cdot = \nu(\tau) - \frac{\varepsilon}{\nu a}[f(\tau, a\sin\psi, a\nu\cos\psi, u) - a\nu'\cos\psi]\sin\psi, \quad \psi(t_0) = \psi_0$$

We pose the problem of changing the oscillation amplitude of system (3.3)

$$a(t_1) = a_1(\tau_1) \qquad (3.4)$$

where a_1 is a given function of τ, greater than zero, while t_1 is not fixed. Here the control performance index is

$$J = \varepsilon \int_{t_0}^{t_1} [k(\tau) + l(\tau)u^2]\,dt \to \min, \qquad k, l > 0 \qquad (3.5)$$

with respect to u. In (3.5) the first term is the "cost" as an amount of time, while the second is the corresponding control resource expenditure.

Further, we examine the case of a function f linear in u

$$f(\tau, x, x^{\cdot}, u) = f_0(\tau, x, x^{\cdot}) + f_1(\tau, x, x^{\cdot})\,u \qquad (3.6)$$

Computing the Hamiltonian function (1.6), from (1.10) we obtain

$$u^* = V(\tau, a, \psi, p, q) = \frac{1}{2l(\tau)\nu(\tau)} f_1(\tau, a\sin\psi, a\nu(\tau)\cos\psi)\left(p\cos\psi - \frac{q}{a}\sin\psi\right) \qquad (3.7)$$

Obviously, u^* is the unique point of maximum of function H with respect to u, which is

$$H^* = \frac{\varepsilon}{\nu}(f_0 - a\nu'\cos\psi)\left(p\cos\psi - \frac{q}{a}\sin\psi\right) + \frac{\varepsilon}{4\nu^2 l} f_1^2\left(p\cos\psi - \frac{q}{a}\sin\psi\right) + \nu q - \varepsilon k \equiv \nu q + \varepsilon h(\tau, a, \psi, p, q)$$

The boundary-value problem is described by Eqs. (3.3) into which expression (3.7) has been substituted, by condition (3.4), and also by the equations for the adjoint variables p and q of type (1.7) and by the transversality conditions

$$p^{\cdot} = -\varepsilon\,\partial h/\partial a, \ p(t_1) = -\alpha; \quad q^{\cdot} = -\varepsilon\,\partial h/\partial\psi, \ q(t_1) = 0$$

The equation of type (1.8), used for determining the process termination instant t_1, is: $h(t_1) = \alpha(t_1)\,a_1'(\tau_1)$, where the prime denotes the derivative with respect to τ_1.

Let us apply the method, developed for constructing the first approximation, in the simple case when there is no dependency on τ, while $t_0 = 0$, $f_0 \equiv 0$, $l = f_1 = 1$. Then on the basis of (2.30) we obtain

$$\xi = -\alpha s / 4\nu^2 + a_0, \quad \eta = -\alpha, \quad s = \varepsilon t, \quad \alpha = 4\nu^2(a_0 - a_1)/s_1$$

The equation for determining the optimal s_1 is: $\alpha^2 = 8\nu^2(k - \nu\varkappa)$; whence, setting $\varkappa = 0$, we find $s_1(0) = \nu\sqrt{2/k}\,|a_0 - a_1|$.

Let us show that the local minimum condition (2.24) is fulfilled, i.e. $ds_1 / d\varkappa\,|_{\varkappa=0} > 0$ or, equivalently, $d\varkappa / ds_1\,|_{s_1=s_1(0)} > 0$. In fact, since $\varkappa = [k - 2\nu^2(a_0 - a_1)^2 / s_1^2] / \nu$,

$$\left.\frac{d\varkappa}{ds_1}\right|_{s_1=s_1(0)} = \frac{4\nu}{s_1^3(0)}\,a_0 - a_1)^2 > 0$$

We also show that the global minimum condition is fulfilled. In fact,

$$\int_{s_1(0)}^{s_1} \chi_1(s_1')\,ds_1' = \frac{k}{s_1}[s_1 - s_1(0)]^2/s_1 \geqslant 0$$

Thus, the approximate solution of problem (3.3)–(3.5) is

$$\xi = -\frac{s}{s_1(0)}(a_0 - a_1) + a_0 = \frac{1}{\nu}\sqrt{\frac{k}{2}}\,\mathrm{sign}\,(a_1 - a_0) + a_0$$

$$u_0^* = \frac{\eta}{2\nu}\cos\psi = \sqrt{2k}\,[\mathrm{sign}\,(a_1 - a_0)]\cos\psi, \qquad \psi = \frac{\nu s}{\varepsilon} + \psi_0 + O(\varepsilon)$$

$$J_0(0) = 2s_1(0)\,k = 2\nu\sqrt{2k}\,|a_0 - a_1|$$

and the "time expenditure" and the control resource expenditure are equal.

As a comparison we consider a similar problem with a constraint on the control: $|u| \leqslant u_0$, while we take time as the performance index: $J = \varepsilon t_1 \to \min$ with respect to $|u| \leqslant u_0$. The solution of the first-approximation problem is

$$\xi = \frac{2}{\pi\nu} u_0 s\,\mathrm{sign}\,(a_1 - a_0) + a_0, \qquad s_1 = \frac{\pi\nu}{2u_0}|a_1 - a_0|$$

$$u_0^* = u_0\,\mathrm{sign}\,[(a_1 - a_0)\cos\psi], \qquad \psi = \frac{\nu s}{\varepsilon} + \psi_0 + O(\varepsilon)$$

Suppose that the process termination time for both the problems posed is the same, i.e. $\nu\sqrt{2/k}\,|a_0 - a_1| = \pi\nu\,|a_1 - a_0|/2u_0$, which corresponds to the value $u_0 = \pi\sqrt{k/2}/2$. Let us compute the control resource expenditure corresponding to control with the u_0 indicated; we obtain

$$\int_0^{s_1} u^2 ds = \frac{\pi^2}{4}\nu\sqrt{\frac{k}{2}}\,|a_1 - a_0|$$

We take the ratio with the expenditure obtained for the performance index (3.5): $u_0^2 s_1 / s_1(0)\,k = \pi^2/8 > 1$. Thus, in the sense indicated, control by a control of type (3.7) proves to be more "economical".

We note that a system of type (3.1) with a fixed control process termination instant $T \sim 1/\varepsilon$ was investigated in [9] for various forms of functionals and of constraints on the control.

Now let $f = f_0(\tau, x, x^{\cdot}) + d(\tau)\,u$ (see [9]); then for index (3.5) we obtain the boundary-value problem

$$\frac{d\xi}{ds} = \frac{1}{\nu} f_{0c} + \frac{1}{4\nu^2}\frac{d^2}{l}\eta, \qquad \xi(s_0) = a_0, \qquad \xi(s_1) = a_1(\tau_1) \tag{3.8}$$

$$\frac{d\eta}{ds} = \frac{1}{2}\frac{\nu'}{\nu}\eta - \frac{1}{\nu}\frac{\partial f_{0c}}{\partial \xi}\eta, \qquad \eta(s_1) = -\alpha, \qquad s = \varepsilon t$$

$$\frac{d\varphi}{ds} = \frac{\nu}{\varepsilon} - \frac{1}{\nu\xi} f_{0s}, \qquad \varphi(s_0) = \psi_0$$

Here

$$\begin{Bmatrix} f_{0c} \\ f_{0s} \end{Bmatrix} = \begin{Bmatrix} f_{0c}(\tau, \xi) \\ f_{0s}(\tau, \xi) \end{Bmatrix} = \frac{1}{2\pi}\int_0^{2\pi} f_0(\tau, \xi\sin\psi, \xi\nu\cos\psi)\begin{Bmatrix} \cos\psi \\ \sin\psi \end{Bmatrix} d\psi$$

The process termination instant s_1 is determined from the relation

$$\frac{1}{\nu}\left(f_{0c} - \frac{\xi\nu'}{2}\right)\eta\Big|_{s_1} + \frac{d^2}{4\nu^2 l}\frac{\eta^2}{2}\Big|_{s_1} = \alpha a_1'(\tau_1) + k$$

For example, let us take $\nu = \mathrm{const}$, $l = d = 1$, $f_0 = -2\lambda x^{\cdot} + \mu x^3$; then, setting $t_0 = 0$, we obtain the exact solution of the averaged boundary-value problem (3.8) as

$$\xi(s, s_1) = a_0 e^{-\lambda s} + \frac{\eta_1}{8\lambda\nu^2} e^{\lambda(s - s_1)}(1 - e^{-2\lambda s}), \qquad \lambda, \mu = \mathrm{const} \tag{3.9}$$

Here

$$\eta_1 = \eta(s_1, s_1) = 8\lambda\nu^2 \frac{a_1 - a_0 e^{-\lambda s_1}}{1 - e^{-2\lambda s_1}}, \qquad \eta(s, s_1) = \eta(s_1, s_1) e^{\lambda(s - s_1)}, \quad a_1 = \text{const}$$

Setting $z = e^{-\lambda s_1}$, we obtain a fourth-order equation for z and under the condition $z > 0$ we find

$$z_{1,2} = \frac{a_0}{2c_{1,2}} + \sqrt{\left(\frac{a_0}{2c_{1,2}}\right)^2 + 1 - \frac{a_1}{c_{1,2}}} \tag{3.10}$$

$$c_{1,2} = a_1 \left(\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{k}{8\lambda^2\nu^2 a_1^2}}\right)$$

Here the plus sign corresponds to a positive η_1, i.e. to an increase in ξ $(a_1 / a_0 > 1)$, while the minus sign, to a decrease in ξ. From the expression obtained it follows that when $a_0 = a_1$ both roots $z_{1,2} = 1$, i.e. $s_1 = 0$. The values of $z_{1,2}$ must satisfy the inequality $z_{1,2} \leqslant 1$, which is equivalent to the nonnegativity of s_1.

Let us investigate the roots $z_{1,2}$ under the assumption that $|a_1 - a_0| \ll a_0, a_1$. In the first approximation with respect to $\Delta a = a_0 - a_1$ we obtain

$$\Delta z_{1,2} \equiv z_{1,2} - 1 \approx \frac{\Delta a}{2c_{1,2} - a_1} = \pm \frac{\Delta a}{a_1 R_1} \approx \pm \frac{\Delta a}{a_0 R_0}, \qquad R_i = (1 + k / 2\nu^2\lambda^2 a_i^2)^{1/2}, \quad i = 0, 1$$

From the expression obtained it follows that $\Delta z_{1,2} \leqslant 0$ for any sufficiently small values of $|\Delta a|$, independently of the sign of Δa, since we should choose the plus sign for negative Δa while a minus sign for positive Δa. Thus, the problem has a solution which is optimal in the sense of (3.5). For the problem of driving $(a_1 > a_0)$ or of damping $(a_1 < a_0)$ the optimal control process termination instants s_1 is

$$s_1 = -\frac{1}{\lambda} \ln z_{1,2} \approx \frac{|\Delta z|}{\lambda} \tag{3.11}$$

The approximate value of the optimal control is

$$u_0^* = \frac{1}{\nu} \eta(s, s_1) \cos\psi, \qquad \psi = \psi_0 + \frac{1}{\varepsilon} \int_0^s \Omega(s')\, ds' + O(\varepsilon) \tag{3.12}$$

$$\Omega(s) = \nu - \frac{3}{8} \frac{\varepsilon\mu}{\nu} \xi^2(s, s_1)$$

where $\Omega = \Omega(s)$ is the perturbed oscillation frequency. Formulas (3.9)–(3.12) yield an approximate solution of the optimal control problem.

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LIAPUNOV CANONIC TRANSFORMATIONS AND NORMAL HAMILTONIAN FORMS

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We investigate to what general form can a Hamiltonian be reduced by an arbitrary canonic transformation preserving the property of Liapunov stability. We have succeeded in answering this question fully in the case of a stable autonomous Hamiltonian. One of the results of the analysis undertaken is a method of reducing the Hamiltonian to normal form in finite order, different from those proposed earlier [1], possessing definite advantages in comparison with them and exposing the connection between the methods of normalization and of averaging. We derive a table allowing us to compute from the original Hamiltonian its third-order normal form in the presence of any third-order resonances. A canonic transformation of the original Hamiltonian to a form more convenient for study is usually used in the investigation of the Liapunov stability of an equilibrium position. From such a viewpoint we can arrive at the method of Birkhoff transformations [2] and many stability results have recently been obtained in this way, having a practical value (for example, [3−5] and others). In the application of the method indicated it is necessary that there exist a close connection between the stability properties of the original and of the transformed Hamiltonians. Therefore, only autonomous transformations are usually used. However, such a restriction is not connected with the conditions for the applicability of the given method even in the case of an autonomous original Hamiltonian. It is interesting to consider this problem from a general point of view, without being tied down to the autonomous case.